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Modification of instability processes by multiplicative noises

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Abstract. We study two dynamical systems submitted to white and Gaussian random noise acting multiplicatively. The first system is an imperfect pitchfork bifurcation with a noisy departure from onset. The second system is a pitchfork bifurcation in which the noise acts multiplicatively on the non-linear term of lowest order. In both cases noise suppresses some solutions that exist in the deterministic regime. Besides, for the first system, the imperfectness of the bifurcation reduces the regime of on-off intermittency. For the second system, the unstable mode can achieve a jump of finite amplitude at instability but without hysteresis. We finally identify a generic property that is verified by the stationary probability density function of the dynamical variable when a control parameter is varied.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.45.-a Nonlinear dynamics and chaos – 91.25.-r Geomagnetism and paleomagnetism; geoelectricity

1 Introduction

Instabilities that occur in a real system are submitted to a certain amount of fluctuations. In some cases these fluctuations are due to external modifications of the system. One can think for instance to oceanic convection and realize that it can be influenced by the evolution of the atmospheric temperature above the ocean. In other cases instabilities are driven by fields that are strongly fluctuating. For the dynamo instability, the motion of an electrically conducting liquid creates magnetic field. The velocity field is the forcing of the magnetic field and in most cases is strongly turbulent at dynamo onset. Turbulent fluctuations thus modify the instability process driven by the time-averaged velocity field (see for instance [1]).

In order to understand the possible effects of these fluctuations, it is natural to study dynamical systems submitted to noise. As a model of system with fluctuating parameter, Graham and Shenzle studied an amplitude equation with noisy departure from onset [2]. In such a case the noise appears multiplying the dynamical variable. Henceforth, one talks about multiplicative noise. The case studied by Graham and Shenzle gained a renewed interest through the work of Yamada et al. and Pikovsky [3]. Numerical simulation of related systems showed a very rich behavior of the dynamical variable that remains close to a weakly unstable manifold for long durations (off-phase) before randomly exploring other states (on-phase). Platt et al. named this phenomena "on-off intermittency" and pointed out its relevance to many instability processes [4]. It has been shown that this phenomena is controlled by the low frequency spectrum of the noisy departure from onset [5].

In the aforementioned works, the noise acts on the departure from onset when the system is close to a supercritical pitchfork bifurcation. In this paper we study two dynamical systems submitted to multiplicative noise. The first is an imperfect pitchfork bifurcation with noisy linear growth rate. The second is a tricritical amplitude equation whose cubic term fluctuates around a fixed value. In both cases, the calculation of the stationary probability density function (PDF) of the variable is a straightforward application of the Fokker-Planck equation. We want to discuss here the possible physical applications of each of these two systems that display interesting behavior. In the first system, one of the deterministic stable solution disappears. Besides on-off intermittency is strongly reduced. In the second system, we show that the noise can lead to a finite jump of the unstable mode at instability onset but with no hysteresis. Based on the behavior of the PDF of the two systems, we finally identify a property of the evolution of the stationary PDF of the dynamical variable when a control parameter is changed.

2 Imperfect pitchfork bifurcation

Several experiments are being performed to achieve the dynamo instability [6,7]. In these experiments, the velocity field is highly turbulent. Therefore an intermittent regime could have been expected close to the instability

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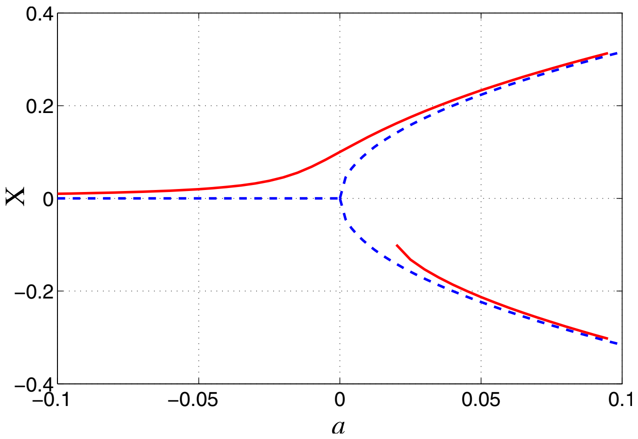

Fig. 1. Stable solution of equation (1) for $\zeta(t) = 0$ and $\epsilon = 0$ (dashed line) and $\epsilon = 0.001$ (full line).

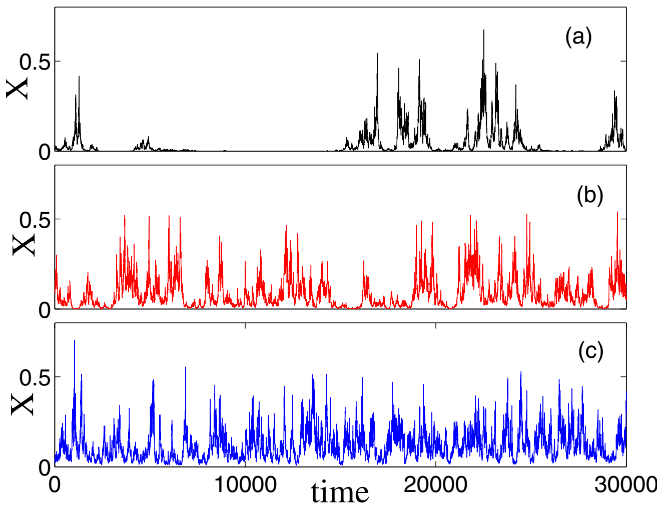

Fig. 2. Temporal traces of the dynamical variable $X(t)$ solution of (1) with $a = 0.01$ and $D = 0.05$ and (a) $\epsilon = 0$; (b) $\epsilon = 0.0001$; (c) $\epsilon = 0.001$.

threshold. However such regime has not been reported for the experiments that have currently reached the dynamo onset [6]. There are several possible explanations. First the spectrum of the turbulent velocity field is far from being white. As mentioned above, when the low frequency spectrum of the departure from onset is reduced, the size of the intermittent regime shrinks [5]. This can explain that no intermittency was observed in the dynamo experiments. Secondly the Earth magnetic field can be put forward. This field acts as a source term in the induction equation that describes the evolution of the magnetic field generated by the dynamo process. For dynamos with strongly constrained velocity field, the fluctuation rate is small and the source term is roughly constant in time. In the following we want to model the likely modification of the dynamo instability process caused by the Earth magnetic field. To estimate qualitatively the effects of this magnetic field we investigate an imperfect pitchfork bifurcation submitted to a multiplicative noise

$$\dot{X} = \epsilon + (a + \zeta(t))X - X^3 \tag{1}$$

where a is the control parameter, $\zeta(t)$ is a random noise and $\epsilon > 0$ quantifies the gap from the perfect pitchfork bifurcation ($\epsilon = 0$).

Without noise, the stable solutions of (1) are shown Figure 1. Algebraic manipulations of third degree equations give the following well-known expression for the solutions.

$$\text{Let} \quad M = \left(\frac{\epsilon}{2} + \sqrt{\frac{\epsilon^2}{4} - \frac{a^3}{27}}\right)^{\frac{1}{3}} + \left(\frac{\epsilon}{2} - \sqrt{\frac{\epsilon^2}{4} - \frac{a^3}{27}}\right)^{\frac{1}{3}}.$$

For any value of the parameters, $x = M$ is a real solution. If $a \geq 3M^2/4$ then two other real solutions are given by $x = -\frac{M}{2} \pm \sqrt{\frac{-3M^2}{4} + a}$, the smaller one being linearly stable.

Figure 2 displays the temporal traces of $X(t)$ for a white and Gaussian random noise defined by

$$\langle \zeta(t)\zeta(t')\rangle = D\delta(t - t'), \tag{2}$$

with $D = 0.05$ and $a = 0.01$, at three different values of ϵ. The increase of ϵ modifies the off-phases that become increasingly fluctuating. Moreover we underline that whatever the initial condition $X(t = 0)$, the system verifies $X(t) \geq 0$ after a transcient behavior. This disappearance of the negative branch in presence of multiplicative noise can be explained qualitively as follows. Starting from a negative value for X, there are events of the forcing that drive the system close to $X = 0$. Then, equation (1) shows that the velocity $\dot{X}$ is equal to ϵ resulting in a drift of X into the half plane $X \geq 0$. Once X is positive it cannot crosses down the X-axis since the velocity $\dot{X}$ is positive for $X = 0$.

These facts are confirmed by the computation of the Probability Density Function (PDF) of X using the Fokker–Planck equation corresponding to (1). Within the framework of the Stratonovich interpretation of white noise, the stationary PDF is [9]:

$$\begin{aligned} &X \leq 0, P(X) = 0 \\ &X > 0, P(X) = C|X|^{2a/D-1} \exp\left(-\frac{X^2}{D} - \frac{2\epsilon}{DX}\right) \end{aligned} \tag{3}$$

where C is a normalization constant.

We show in Figure 3 the theoretical prediction (3) and the PDF extracted from the numerical simulations displayed in Figure 2. The effect of ϵ is to induce a cut-off in the PDF for $X_c \simeq 2\epsilon/D$. Below this value, the PDF tends fastly towards zero. For $2a < D$, the PDF can still follow a power law with negative exponent for

$$\frac{2\epsilon}{D} \ll X \ll \sqrt{D}. \tag{4}$$

The width of this power law is strongly reduced when ϵ increases. This is the signature of the disappearance of the intermittent regime when ϵ increases as displayed in Figure 2. Indeed, a positive ϵ forces X to wander away

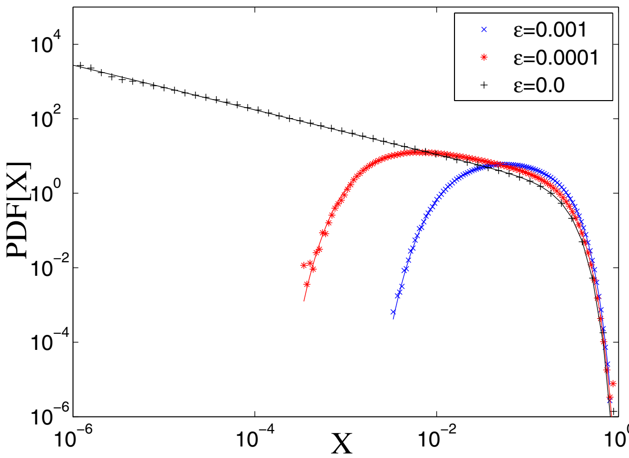

Fig. 3. PDF of the dynamical variable given by (1) with: $a = 0.01$, $D = 0.05$ and $\epsilon = 0$ (+); $\epsilon = 0.0001$ (∗); $\epsilon = 0.001$ (×). Full lines correspond to the equation (3).

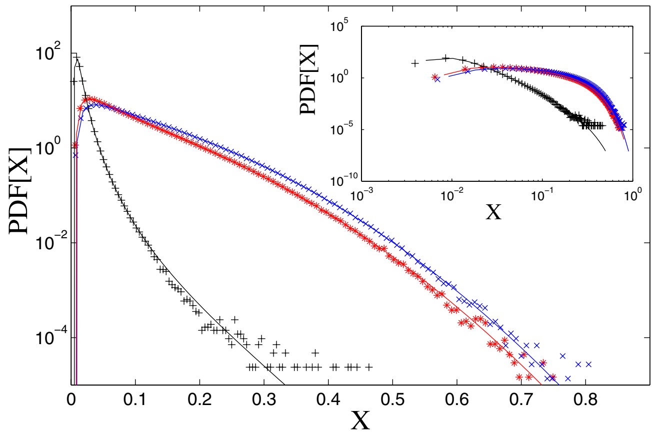

Fig. 4. PDF of the dynamical variable given by (1) with $\epsilon = 0.001$, $D = 0.05$ and: $a = -0.001$ (×); $a = -0.01$ (∗); $a = -0.1$ (+). Full lines correspond to the equation (3). The inset is the same figure in log-log scales.

from $X = 0$. Even in the presence of noise, the system does not spend long durations close to $X = 0$. Consequently its PDF does not diverge for $X = 0$. Besides, for high enough ϵ (of the order of a fraction of $D^{3/2}$ or larger), the PDF does not display any power law. Notice that for a perfect supercritical bifurcation ($\epsilon = 0$) a similar reduction of the on-off intermittent behavior can also be due to additive noise [8].

Finally we underline that the PDF (3) is defined for all values of a — positive or negative — i.e. there are extended fluctuations even for $a \ll 0$ as illustrated in Figure 4. Therefore the threshold is smoothed and less precisely defined than in the perfect case ($\epsilon = 0$) where the PDF of X is a delta function for $a < 0$. This is because, even for negative a, $X = 0$ is not a solution for non-zero ϵ. Therefore the noisy term $\zeta(t)X(t)$ does not vanish as it would do if $X = 0$ were a solution. The effect is then similar to that of an additive noise that enlarges the PDF of the stationary state.

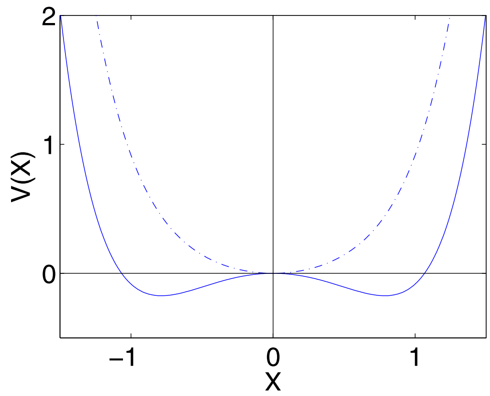

Fig. 5. Potential $V(X)$ defined by equation (6) for $b = -1$, (−): $a = 1$, (−·): $a = -1$.

3 Amplitude equation with a fluctuating non linear term

We now study an amplitude equation with a fluctuating cubic term

$$\dot{X} = aX + (b + \zeta(t))X^3 - X^5, \tag{5}$$

where X is a function of time t and $\zeta(t)$ is a noise.

We first discuss the well-known no-noise regime, $\zeta(t) = 0$. Equation (5) can be seen as the evolution equation of an overdamped particle moving in the potential

$$V(X) = -\frac{aX^2}{2} - \frac{bX^4}{4} + \frac{X^6}{6}. \tag{6}$$

If $b < 0$, both non-linear terms limit the linear growth that occurs for positive a. We plot in Figure 5 the form of V for a positive and a negative value of a and $b = -1$.

The instability is supercritical. It does not display hysteresis and, in Figure 1, the asymptotic solution of equation (5) as a function of a is presented (dashed line). For small values of a, the long time value of X is $\pm\sqrt{-a/b}$ depending on the initial condition.

If $b = 0$, there is no cubic term. The instability is tricritical. It does not display hysteresis and for positive a the long-time value of X is $\pm a^{1/4}$ depending on the initial condition.

If $b > 0$, the cubic term can lead to a non-linear instability. We plot in Figure 6 the form of V for different values of a, b being fixed to one. If a is smaller than $-b^2/4$, V has only one minimum at $X = 0$. If $-\frac{b^2}{4} \leq a \leq 0$, there are three minima: $X = 0$, $X = \pm X_m$ with $X_m = (\frac{b+\sqrt{b^2+4a}}{2})^{1/2}$. For $a \geq 0$, there are two minima $X = \pm X_m$. The instability is subcritical and displays hysteresis for $-\frac{b^2}{4} \leq a \leq 0$. We plot in Figure 7 the long time value of X.

We now tackle the case with noise. ζ is chosen to be a Gaussian white noise with autocorrelation given by equation (2). Equation (5) is understood as a Stratonovich equation [9] and we calculate the expression of the stationary PDF of X. We obtain for negative a,

$$P(X) = \delta(X),$$

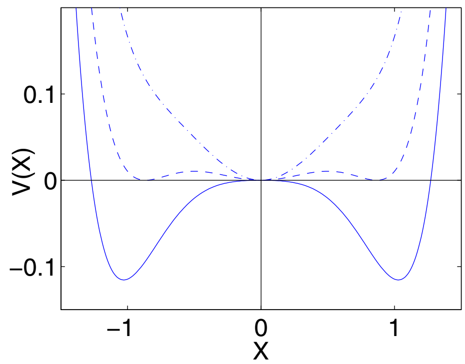

Fig. 6. Potential $V(X)$ for $b = -1$, (−·): $a = -1/2$, (−−): $a = -3/16$, (−): $a = 1/16$.

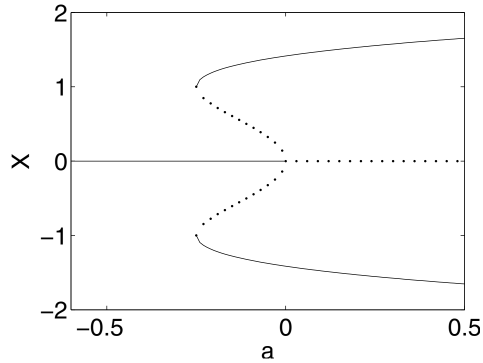

Fig. 7. Asymptotic value of X for $b = 1$, (−) stable solutions, (.) unstable solutions.

and for positive a,

$$P(X) = C\, X^{-\frac{2}{D}-3} \exp\left(-\frac{a}{2D\, X^4} - \frac{b}{D\, X^2}\right), \qquad (7)$$

where C is a normalisation constant. We first note that the solution $X = 0$ is unstable for positive a regardless of the value of b.

We now look for the most probable value X_{mp} of X. It is a solution of

$$(2 + 3D)X_{mp}^4 - 2bX_{mp}^2 - 2a = 0\,. \qquad (8)$$

In the small a limit, we get

$$\begin{aligned} b < 0,\ X_{mp} &\simeq \left(\frac{a(2+3D)}{|b|D}\right)^{\frac{1}{2}}, \\ b = 0,\ X_{mp} &= \left(\frac{2a}{2+3D}\right)^{\frac{1}{4}}, \\ b > 0,\ X_{mp} &\simeq \frac{2b}{2+3D}. \end{aligned} \qquad (9)$$

Concerning the moments of X, we note that the PDF is large and that no moment $\langle X^n \rangle$ exists for $n \geq 2 + 2/D$. In any case, the second moment exists and in the small a limit reads

$$\begin{aligned} b < 0,\ \langle X^2 \rangle &\simeq \frac{a}{|b|}, \\ b = 0,\ \langle X^2 \rangle &\simeq \left(\frac{a}{2D}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2D})}{\Gamma(\frac{1}{2D}+\frac{1}{2})}, \\ b > 0,\ \langle X^2 \rangle &\simeq b. \end{aligned} \qquad (10)$$

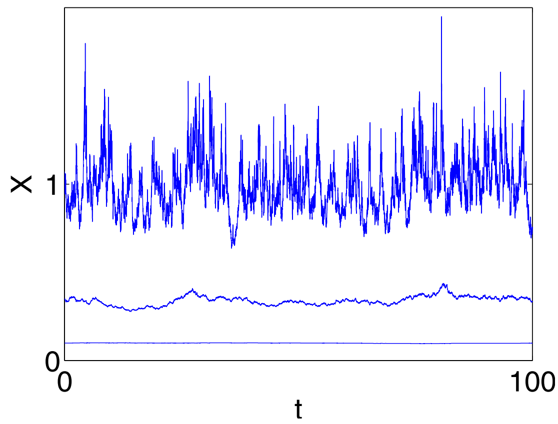

Fig. 8. Temporal traces $X(t)$ solution of equation (5) for $D = 0.1$, $a = 0.01$ and from the bottom to the top $b = -1$, $b = 0$ and $b = 1$.

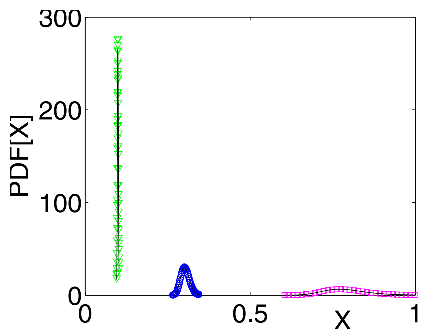

Fig. 9. Stationary PDF of X solution of equation (5) for $b = -1$, $D = 0.1$ and (□): $a = 1$, (○): $a = 0.1$ and (▽): $a = 0.01$. The related theoretical prediction, equation (7), are displayed as continuous line.

To test our predictions, we solve numerically equation (5). Note that if $X(t = 0)$ is positive, resp. negative, then X remains positive for all time, resp. negative. Therefore we restrict our calculations to positive X. We display in Figure 8 the solution for $a = 0.01$, $D = 0.1$ and $b = -1, 0, 1$. We plot in Figures 9, resp. 10, the stationary PDF of X for $b = -1$, resp. $b = 0$, and various positive values of a. The second moments of X are displayed in Figures 11, 12.

In the case $b \leq 0$, the noise does not strongly modify the instability process. It merely changes the coefficients of the laws that relate X_{mp} and $\langle X^2 \rangle$ to a. The noise does not change the exponent of these laws: X_{mp} and $\sqrt{\langle X^2 \rangle}$ remain proportional to $a^{1/2}$ for $b < 0$ and to $a^{1/4}$ for $b = 0$ as in the deterministic case.

The effect of the noise is more important for $b > 0$. Indeed the hysteretic behavior of the deterministic system

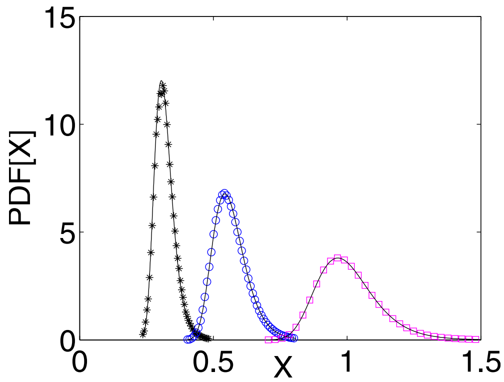

Fig. 10. Stationary PDF of X for $b = 0$, $D = 0.1$ and ($\Box$): $a = 1$, ($\circ$): $a = 0.1$ and ($*$): $a = 0.01$. The related theoretical prediction, equation (7), are displayed as continuous line.

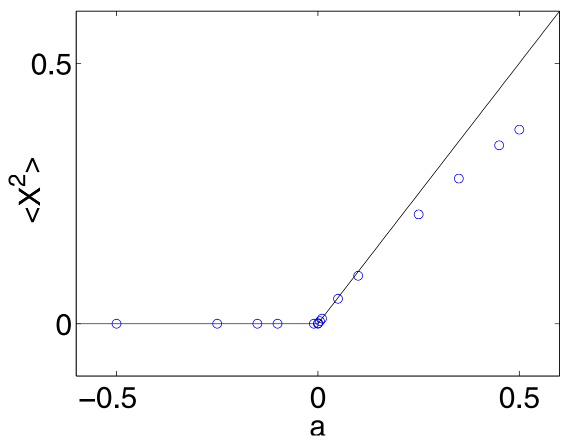

Fig. 11. Second moment of X, $\langle X^2 \rangle$, as a function of a for $b = -1$ and $D = 0.1$. (o): numerical simulations, ($-$) theoretical prediction equation (10).

is destroyed. In the presence of noise, the instability takes place at $a = 0$. Both X_{mp} and $\langle X^2 \rangle$ realize a jump of finite amplitude at $a = 0$. They are discontinuous but without hysteresis. We plot in Figure 13 the numerically computed PDF of X for $b = 1$ and different values of a. The value of the second moment $\langle X^2 \rangle$ is displayed as a function of a in Figure 14.

This is the main effect of the noise acting on the cubic term: for positive b, it leads to a discontinuous instability but with no hysteresis. The disappearance of the subcritical branch can be explained as follows. For positive b and $-b^2/4 \leq a \leq 0$, the potential $V(X)$ has three minima. The minimum at $X = 0$ is locally stable whatever the sign of b. Thus if X gets close to zero, it remains there. The other two minima are locally stable for positive b. However the coefficient of the cubic term is $b + \zeta(t)$. It fluctuates and, from time to time, is negative. Then the two non-zero minima disappear and X tends to the only remaining minimum $X = 0$. If X gets close enough to zero before the two other minima appear again, then it remains there because $X = 0$ is locally stable. Note that the time required for X to escape from one of the non-zero minima diverges when a tends to zero by negative values. Numerical simulations require careful estimation of this duration to achieve stationary states.

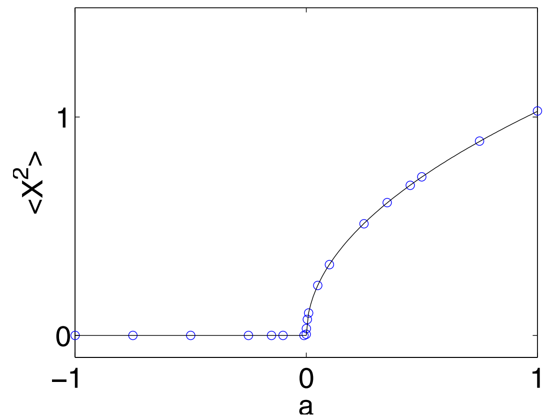

Fig. 12. Second moment of X, $\langle X^2 \rangle$, as a function of a for $b = 0$ and $D = 0.1$. (o): numerical simulations, ($-$) theoretical prediction equation (10).

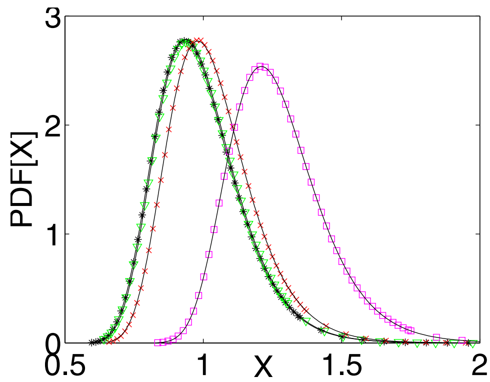

Fig. 13. Stationary PDF of X for $b = 1$, $D = 0.1$ and ($\Box$): $a = 1$, ($\times$): $a = 0.1$, (∇): $a = 0.01$ and ($*$): $a = 0.001$. The related theoretical prediction, equation (7), are displayed as continuous line.

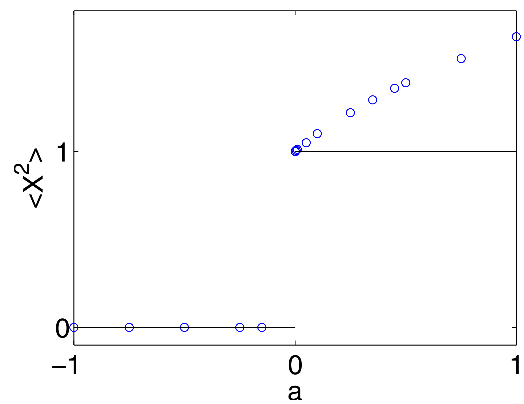

Fig. 14. Second moment of X, $\langle X^2 \rangle$, as a function of a for $b = 1$ and $D = 0.1$. (o): numerical simulations, ($-$) theoretical prediction equation (10).

Horsthemke and Malek-Mansour studied a similar instability process submitted to multiplicative noise [10]. By solving the Fokker-Planck equation related to their model, they predicted the disappearance of a subcritical branch. Our explanation for the mechanism governing this

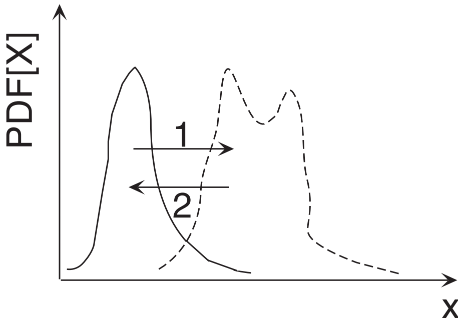

Fig. 15. Sketch of the generic processes involved in the modifications of a PDF. Process 1 is the creation of a new maximum, process 2 is the disappearance of an existing maximum.

phenomenon is also pertinent for their model. Moreover, in our case, we study the whole bifurcation diagram of the instability. We prove that the disappearance of the subcritical branch can lead to a discontinuous behavior for any moment of the variable but without hysteresis.

4 Summary

We now summarize the results obtained for the two systems under study. In both cases, the white and Gaussian multiplicative random noise can suppress solutions that exist in the no-noise regime. Other effects depend on the system.

For an imperfect pitchfork bifurcation with noisy linear growth rate, we identify three main effects. First, if $\epsilon > 0$, the stationary state verifies $X(t) \geq 0$. Therefore one of the deterministic branches of solution disappears. Any amount of noise leads to the suppression of this branch. Secondly, even for negative a, the PDF is not a delta-function. Therefore multiplicative noise acting on an imperfect pitchfork bifurcation leads to large PDF whatever the parameter values. Third, the power-law with negative exponent of the PDF and the on-off intermittent behavior of the system are reduced by an imperfect bifurcation. This can be important in experiments that try to achieve on-off intermittency.

For an amplitude equation with fluctuating cubic term the effect of noise depends on the average of the coefficient of the cubic term. If it is in average negative or zero, i.e. if without noise the instability is supercritical or tricritical, the effect of noise is just to modify the coefficient of the laws that relate the second moment $\langle X^2 \rangle$ or the most probable value X_{mp} with the departure from onset a. The effect is more striking if b is positive, i.e. if without noise, the instability were subcritical. Then the subcritical branch disappears. The solution is such that $\langle X^2 \rangle$ or X_{mp} achieves a jump of finite amplitude at instability onset (for $a = 0$) but without hysteretic behavior. Notice that the jump occurs at $a = 0$ and is therefore located at an higher value than that of the "Maxwell plateau" defined by the two minima of the potential having the same value.

To conclude, we present a conservation law that is verified by the stationary PDF of a variable when a control parameter is changed. Let $P_a(X)$ be the stationary PDF of a random variable X. The average on all the possible initial conditions is performed and a is a parameter that controls the dynamical evolution of the variable X. We define the index $\mathcal{A}$ of this PDF by

$$\mathcal{A} = \sharp \min P_a - \sharp \max P_a, \tag{11}$$

i.e. the difference between the number of local minimum of P_a and the number of local maximum of P_a. We believe that $\mathcal{A}$ is a constant that does not depend on a. We do not have rigorous proof of this result but a heuristic one is based on the following idea. When a PDF changes shape because of the variation of a parameter, only two behavior are possible. These processes are sketched in Figure 15. First a new maximum of the PDF can appear. Simultaneously a new minimum is created between the former maximum and the new one. Secondly the reverse process can take place: two maxima can merge into one maximum. Then one maximum disappears but simultaneously the minimum that was located between the two maxima disappears. Therefore, for both processes the index $\mathcal{A}$ of the PDF is conserved. If there are no other possible process of evolution of the PDF, then $\mathcal{A}$ is constant. Note that for the processes studied here and in other works [2,5], $\mathcal{A}$ is indeed a constant.

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